N-Queens Problem

Latin Squares

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The N-Queens Problem

The N-Queens problem originates from a question relating to chess, The 8-Queens problem. Chess is played on an 8×8 grid, with each piece taking up one cell. A queen is a piece in chess that, in any given move, can move any distance vertically, horizontally, or diagonally. However, the queen cannot move more than one direction per turn. It can only move one direction per turn. So a question one might ask themselves is whether or not you can place 8 queens on a chessboard so that none of the queens can kill each other in one move (i.e. There is no way for a queen to in one cell to reach a queen in another cell in one move)? The answer is yes!
Here is one way this can be achieved:

Q = Queen

					Q		
			Q				
						Q	
Q							
							Q
	Q						
				Q			
		Q					

There are 12 unique solutions to this problem. Two solutions are not unique if you can "mirror" one solution to find the other, or if you can rotate the board to find the other solution, or a combination of the two moves.

We can generalize the 8-Queens problem to be the N-Queens problem.

Question. Given an $n \times n$ chessboard, can we place n queens on the chessboard so that none of the queens can kill each other in one move?

Example.
$n = 4$:

		Q	
Q			
			Q
	Q		

One may think we can find solutions for all values of n, upon trying a few small values of n, we find that no solutions exist for $n = 2$ or $n = 3$. Here we see why:

$n = 2$:
The only option we have is to place a queen in the corner of board.

Q	

No matter where we put the first queen, it can always move to any other spot on the board in one move. Thus, we cannot place a second queen to satisfy the N-Queens problem.

$n = 3$:
This doesn't work for a reason similar to why $n = 2$ doesn't work. If we place a queen in the center of the board, then it can always move to any other spot on the board in one move.

	Q	

So placing a queen at the center of the board does not let us place a second queen. So we can try placing a queen in the corner of the board.

Q		

Now we have to cells that the queen cannot reach in one move. So we can now place a second queen.

Q		
	Q	

However, putting a queen in any of those two cells makes it so that both queens can reach any other cell on the board. Therefore we do not have the option to place a third queen. Any place we put the first queen leaves (at most) two other places to put queens without conflict with the first queen, but the 2nd and 3rd queens will always be able to reach each other in one move, so $n = 3$ cannot satisfy the N-Queens problem.

The N-Queens problem seems inherently similar to concepts of Latin Squares. It turns out Latin Squares are in fact useful when working with the N-Queens problem!

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Definition. A **broken right diagonal** of Latin square L is the set of n cells that start from choosing any cell in the top row, then taking the cell that is in the row below and one cell to the right. Repeat this process until we reach the last row. If the cell reaches the last column, then wrap around to the first column of the next row. A broken right diagonal is sometimes called a wrap around right diagonal.
Example.

			○	
				○
○				
	○			
		○		

Definition. A **broken left diagonal** is the same as a broken right diagonal, except we wrap around to the left instead of the right.
Example.

		○		
	○			
○				
				○
			○	

The concept of broken diagonals will give us the special type of Latin square that will help us find solutions to the n-queens problem.

Definition. A Latin square L is called **pandiagonal** if there are no repeated symbols in any broken diagonal. I.e., every symbol appears exactly once in each broken diagonal.
Example.

0	1	2	3	4
3	4	0	1	2
1	2	3	4	0
4	0	1	2	3
2	3	4	0	1

None of the broken diagonals have any repeats, therefore we have a pandiagonal Latin square.

So now take a pandiagonal Latin square L. We know that L does not have repeated symbols in any row or column because it is a Latin square. We also know that L does not have repeated symbols in its broken diagonals because it is a pandiagonal Latin square. Therefore, if we choose any symbol s in our Latin square, and place a queen on top of each cell that contains an s, then none of the queens will be able to capture each other. So we will have placed n queens on an $n \times n$ grid so that none of them can capture each other. So, if a pandiagonal Latin square of order n exists, then we know we can find a solution to the n-queens problem for that order. Pandiagonal Latin squares actually give us n (not necessarily distinct) solutions to the n-queens problem. But, you may find yourself asking the following question:

Question. For what order do pandiagonal Latin squares exist?

Well, consider the following construction for L:

Take any $a, b \in \{0, 1, \ldots, n-1\}$. Let the n^2 cells of L be represented by the following grid

$L =$

0	a	$2a$	$3a$	$\ldots$	$(n-1)a$
b	$b+a$	$b+2a$	$b+3a$	$\ldots$	$b+(n-1)a$
$2b$	$2b+a$	$2(b+a)$	$2b+3a$	$\ldots$	$2b+(n-1)a$
$3b$	$3b+a$	$3b+2a$	$3(b+a)$	$\ldots$	$3b+(n-1)a$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$(n-1)b$	$(n-1)b+a$	$(n-1)b+2a$	$(n-1)b+3a$	$\ldots$	$(n-1)(b+a)$

mod n

In general, $L_{i,j}$ (which denotes the cell in the ith row and jth column) is $ai + bj \mod n$.

Claim 1:

In order for L to be a Latin square, a, b must all be relatively prime to n.

Proof. Since L is Latin square, we know there is no symbol that is repeated in any row. In other words, there are no two symbols $L_{i,j}$ and $L_{i,k}$ that are the same. Notice that, by our construction, this only happens if $ai + bj \equiv ai + bk \mod n$.

$\Rightarrow bj \equiv bk \mod n$. If b and n share a factor, let us call it f, and we set $j = 0$, and $k = \frac{n}{f}$ then we have $bj = 0$, and bk will be a multiple of n. Therefore $bk \equiv 0 \mod n$, so $bj \equiv bk \mod n$. So b and n cannot share a factor. If b and n are relatively prime, then b will have an inverse $\mod n$. Therefore if we look at $b^{-1}bj \equiv b^{-1}bk \pmod{n}$, then we know that j must be congruent to k because $b^{-1} \cdot b \equiv 1 \pmod{n}$. So $bj \equiv bk \pmod{n}$ will only happen when $j = k$, i.e., $L_{i,j}$ and $L_{i,k}$ are the same cell.

Similarly, if we look at the columns of L, we will see that a and n must also be relatively prime by the same argument. □

Claim 2:

In order for L to be a pandiagonal Latin square, $a + b$ and $a - b$ must be relatively prime to n.

Proof. Take any broken right diagonal in L. Since $L_{i,j}$ is of the form $ai + bj$, the entry that is below and to the right of $L_{i,j}$, i.e. $L_{i+1,j+1}$, is of the from $a(i+1) + b(i+1) =$

$ai + bj + a + b = L_{i,j} + a + b$. With this observation, it is easy to see that any broken right diagonal is of the following form:

	$\dots$	ka		$\dots$	
	$\dots$		$ka + (a+b)$	$\dots$	
	$\vdots$			$\ddots$	
	$\dots$			$\dots$	$ka + (n-k-1)(a+b)$
$ka + (n-k)(a+b) \dots$					
	$\ddots$				
	$\dots$	$ka + (n-1)(a+b)$			

So if $a + b$ is relatively prime to n, Then each multiple of $(a + b)$ will be distinct mod n. Therefore each entry in our broken right diagonal will be distinct. Similarly, if $a - b$ is relatively prime to n, each entry in our broken left diagonal will be distinct. Therefore, if $a + b$ and $a - b$ are relatively prime to n, then L is a pandiagonal Latin square.

□

In a perfect world, this construction for pandiagonal Latin squares would exist for every order. Sadly, we do not live in such a world. They do not exist for even orders or an orders that are divisible by 3. Before we begin that proof, we must look at the following definition first.

Definition. A **transversal** of a Latin square is a way to pick out n symbols so that you have one from each row, each column, and each symbol is represented.

Example.

3	1	4	2
4	2	3	1
2	4	1	3
1	3	2	4

Observation. If a Latin square L has an orthogonal mate, then L can be divided into n distinct transversals.

Proof. Take a Latin square L and its orthogonal mate M. Choose any symbol s in M. Look at the cells containing s in M. Since L and M are orthogonal, we know that these same cells in L cannot have any repeated symbols. Since L and M are Latin squares, we know that these symbols do not repeat in any row or column. Therefore each symbol s that appears in M corresponds to a transversal in L. Since we have n distinct symbols, we have n distinct transversals. □

Corollary. If L is a Latin square of a cyclic group (for example, $\mathbb{Z}/n\mathbb{Z}$) of even order, then L does not have an orthogonal mate.

Claim 3:
Pandiagonal Latin squares do not exist for even orders.

Proof. Assume L is a pandiagonal Latin square of even order. Look at the following Latin square M:

$$M = \begin{array}{|c|c|c|c|} \hline 0 & 1 & \dots & n-1 \\ \hline 1 & 2 & \dots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline n-1 & 0 & \dots & n-2 \\ \hline \end{array}$$

Notice that the broken left diagonals of M are the same symbol repeated. We also know that the broken left diagonals of L have no repeating symbols, since L is a pandiagonal Latin square. Therefore L and M are mutually orthogonal, because each pair $(L_{i,j}, M_{i,j})$ will appear only once. However, our corollary tells us that M cannot have an orthogonal mate because it is a Latin square of a cyclic group of even order. Therefore we have a contradiction, and L cannot be a pandiagonal Latin square of even order. □

Pandiagonal Latin squares also do not exist for orders of n where n is a multiple of 3. We have tried several proof methods, but have yet to succeed in proving this claim.

Although pandiagonal Latin squares may not exist for every order, the fact that a connection between the n-queens problem and Latin squares exists is very exciting! (It also suggests that Latin squares may be the coolest structures in existence. Stay tuned for a proof of this claim). However, the interesting thing about the n-queens problem is not whether or not a solution exists, because it is known that solutions exist for $n \geq 4$. What is interesting is the number of distinct solutions that exists. The number of different solutions, not necessarily distinct, is known for $n \leq 26$.

n	Number of different solutions
1	1
2	0
3	0
4	2
5	10
6	4
7	40
8	92
9	352
10	724
11	2680
12	14200
13	73712
14	365596
15	2279184
16	14772512
17	95815104
18	666090624
19	4968057848
20	39029188884
21	314666222712
22	2691008701644
23	24233937684440
24	227514171973736
25	2207893435808352
26	22317699616364044

The number of distinct solutions is also known for $n \leq 26$.

n	Number of distinct solutions
1	1
2	0
3	0
4	1
5	2
6	1
7	6
8	12
9	46
10	92
11	341
12	1787
13	9233
14	45752
15	285053
16	1846955
17	11977939
18	83263591
19	621012754
20	4878666808
21	39333324973
22	336376244042
23	3029242658210
24	28439272956934
25	275986683743434
26	2789712466510289

It is clear to see that the number of solutions grows exponentially. However there is an odd case. There are more solutions for $n = 5$ than for $n = 6$. This is the only case where something like that happens, but it is very interesting. The number of solutions that exist when $n \geq 27$ is still an open problem. But other people have also looked into different variations of the n-queens problem. For example, given an $n \times n$ board, what is the smallest number of queens one can place (denoted by $s(n)$) so that they cover the entire board, yet none of them can capture each other? Here are the values for $n \leq 8$:

n	$s(n)$	Number of different solutions
1	1	1
2	1	1
3	1	1
4	3	2
5	3	2
6	4	17
7	4	1
8	5	91

Another spin-off from the n-queens problem is the n-rooks problem. A rook is a chess piece that can only move vertically and horizontally. So it is like a queen, except it cannot move along the diagonals. It is actually known that there are $n!$ solutions to the n-rooks problem, which is really cool!

Now, you may find yourself wondering how someone found all the different possible solutions for the n-queens problem for $n \leq 26$. Well, the most popular method is with what is called a backtracking program. The backtracking program runs the following algorithm to find a solution to the n-queens problem:

- Place a queen in the top row, and keep track of the column and diagonal that it occupies
- Place a queen in the next row down, avoiding the same column and diagonal as the queen above, and keep track of the column and diagonal it occupies
- Move on to the next row
- If no position is available in the next row, back track to the previous row and move the queen to the next available spot
- Continue this process until a queen is placed on each row

The number of different solutions for $n = 26$ was found in July of 2009. The number of different solutions for $n = 27$ is still unknown, but who knows, maybe it will be in the quintillions.

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